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LETTER TO THE EDITOR

Intrinsic fluctuations associated with the onset of a centre manifold

Ariel Fernández
Frick Laboratory, Princeton University, Princeton, NJ 08544, USA

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Abstract. A stochastic centre manifold treatment is applied to study a subcritical inverted bifurcation scenario. We demonstrate that the intrinsic fluctuations are subject to scaling relations involving the small parameters characteristic of the particular unfolding. A stochastic counterpart of the Lorenz system reduced to Poincaré normal form is analysed.

After the pioneering work by Ruelle and Takens [1], the relevance of centre manifold (CM) reductions became apparent when studying the onset of strange attractors in hydrodynamic systems.

Although the Ruelle–Takens scenario and the subcritical inverted bifurcation scenario for the transition to turbulence have been extensively investigated (see, for example, [2–4]), the nature and effect of intrinsic fluctuations in hydrodynamic critical phenomena remains obscure [5, 6]. The stochastic counterparts of the relevant order parameter equations are poorly understood [5, 6].

A starting point for studying this problem is the fact that, in a neighbourhood of a critical point, the probability distribution of fast-relaxing degrees of freedom is confined to a narrow strip along the CM [7, 8]. The slowly varying degrees of freedom are the order parameters of the system which, in this letter, will be regarded as the CM coordinates [8].

To narrow down the scope of our discussion we shall concentrate on a stochastic counterpart of the subcritical inverted bifurcation scenario for the onset of a Lorenz attractor [4]. As we shall show in this letter, in this particular case the width of the strip depends on the slowly varying order parameters of the system, i.e. on the position on the CM.

The main result of this letter is to establish the fact that the proper scaling of the random source terms cannot be obtained by extrapolating from results on thermal equilibrium fluctuations [6] but depends on the small parameters characteristic of the instability. Thus we can state that *the scaling of the fluctuations which lead to the onset of a CM representing a dissipative structure is part of the information already contained in the Poincaré normal form associated with the CM.*

It should be emphasised that, unlike the case of a Hopf bifurcation, the CM, being a locally attractive and locally invariant surface, does not contain the limit cycle. This cycle is unstable and appears below the critical value (r_T) for the bifurcation parameter, i.e. $r < r_T$. Instead, the CM is tangent at the laminar steady state for $r = r_T$ to the space spanned by the eigenmodes with neutral stability. These modes are associated with the eigenvalues which cross the imaginary axis at criticality [7, 8], i.e. the CM coincides locally with the Lorenz-invariant surface which emerges beyond the dynamic instability and corresponds to the sudden transition to turbulent behaviour [4].

The object of interest in our approach is the probability density functional $P = P(\boldsymbol{X}_s, \boldsymbol{X}_f, t)$. The components of the vector $\boldsymbol{X}_s$, $X_{s,i}$, $i = 1, \ldots, S$, are the slowly varying order parameters or CM coordinates of the system. The components of $\boldsymbol{X}_f$ are the fast-relaxing degrees of freedom $X_{f,j}$, $j = 1, \ldots, F$, which are enslaved by the order parameters in a neighbourhood of the critical point. This statistical subordination justifies the following ansatz [7, 8]:

$$P(\boldsymbol{X}_s, \boldsymbol{X}_f, t) = \tilde{P}(\boldsymbol{X}_s, t) Q(\boldsymbol{X}_f | \boldsymbol{X}_s) \tag{1}$$

where the factor Q represents a conditional probability and is given by

$$Q(\boldsymbol{X}_f | \boldsymbol{X}_s) = \prod_{j=1}^{F} (g_j/\pi)^{1/2} \exp\{-g_j[X_{f,j} - \tilde{F}_j(\boldsymbol{X}_s)]^2\}. \tag{2}$$

In this relation, $X_{f,j} = \tilde{F}_j(\boldsymbol{X}_s)$ is the CM equation and the Gaussian width $d_j = (2g_j)^{-1/2}$ is a function of the position on the CM as will be proven in this letter:

$$d_j = 2^{-1/2}\left(g_{j0} + \sum_{i=1}^{S} g_{ji} X_{s,i}\right)^{-1/2}. \tag{3}$$

The problem of fluctuation scaling reduces to the problem of finding the adequate scaling factor for the g_{ji} so that the Fokker-Planck (FP) equation for P can be integrated with respect to the $X_{f,j}$ along the CM providing an equation of continuity for $\tilde{P}$, i.e. allowing for a continuous flow of probability about the CM.

The reduced FP equation for $\tilde{P}$ should correspond therefore to the stochastic counterpart of the CM-reduced Lorenz system expressed in Poincaré normal form. Let $\mathbf{T}$ represent the transformation associated with the local CM reduction of fast-relaxing variables $\mathbf{T}: \boldsymbol{X} \rightarrow (\boldsymbol{X}_s, \boldsymbol{X}_f)$. Then the reduced FP equation is obtained by making use of (1)-(3) and computing the following integral:

$$\begin{aligned}\int_{CM} \partial_t P(X_s, X_f, t)\, dX_f = \int_{CM} \Bigg(&-\sum_{i=1}^{S} \partial_{X_{s,i}}\{[\dot{X}_{s,i} - (\mathbf{T}f)_{s,i}]P\} - \sum_{j=1}^{F} \partial_{X_{f,j}}\{[\dot{X}_{f,j} - (\mathbf{T}f)_{f,j}]P\} \\ &+ \sum_{i,i'=1}^{S} \tfrac{1}{2}\tilde{C}_{ii'} \partial^2_{X_{s,i}X_{s,i'}} P + 2\sum_{i=1}^{S}\sum_{j=1}^{F} \tfrac{1}{2}\tilde{C}_{ij} \partial^2_{X_{s,i}X_{f,j}} P + \sum_{j,j'=1}^{S} \tfrac{1}{2}\tilde{C}_{jj'} \partial^2_{X_{f,j}X_{f,j'}} P \Bigg) dX_f\end{aligned} \tag{4}$$

where $f = f(t)$ is the random source for intrinsic fluctuations, $\tilde{C}_{ij}$ are fluctuation covariances:

$$\tilde{C}_{ij}\delta(t - t') = \langle (\mathbf{T}f)_{a,i}(t)(\mathbf{T}f)_{b,j}(t') \rangle \qquad a, b = \mathrm{s, f} \tag{5}$$

and the averages are taken over an ensemble of realisations of $\mathbf{T}f(t)$.

Specifically, we shall consider the Lorenz equations

$$\begin{aligned} \dot{X} &= \sigma(Y - X) \\ \dot{Y} &= -X(Z - r) - Y \\ \dot{Z} &= XY - bZ \end{aligned} \tag{6}$$

where r is the reduced Rayleigh number ($r = R/R_c$ with R_c the critical value for the onset of the laminar convective state), σ is the Prandtl number and b is a geometrical ratio [4].

The laminar state is $(A, A, r-1)$ with $A=[b(r-1)]^{1/2}$, $\sigma>b+1$, $b>0$ and the critical value of the bifurcation parameter is $r=r_T=(\sigma+b+3)/(\sigma-b-1)$.

The inverse transformation $\mathbf{T}^{-1}$ (see [9] for details) is given by

$$\mathbf{T}^{-1}=\begin{bmatrix} 2 & 0 & 1 \\ 2 & -2w/\sigma & 1+\dfrac{\lambda_3}{\sigma} \\ \dfrac{2w^2}{\sigma A} & \dfrac{2w(\sigma+1)}{\sigma A} & \dfrac{-\lambda_3(1+\sigma+\lambda_3)}{\sigma A} \end{bmatrix} \tag{7}$$

$$\mathbf{T}^{-1}:(X_{s,1}, X_{s,2}, X_{f,1})\rightarrow(X-A,\ Y-A, Z-(r-1)). \tag{8}$$

Since we have only two order parameters and one subordinated degree of freedom, the notation can be simplified as follows:

$$X_{s,1}=X_1 \qquad X_{s,2}=X_2 \qquad X_{f,1}=X_3 \qquad g_j=g. \tag{9}$$

The system transformed under $\mathbf{T}$ is now in Poincaré normal form:

$$\begin{aligned} \dot{X}_1&=-wX_2+a_1X_1^2+a_2X_1X_2+a_3X_1X_3+a_4X_2X_3+a_5X_3^2 \\ \dot{X}_2&=wX_1+b_1X_1^2+b_2X_1X_2+b_3X_1X_3+b_4X_2X_3+b_5X_3^2 \\ \dot{X}_3&=\lambda_3X_3-2(a_1X_1^2+a_2X_1X_2+a_3X_1X_3+a_4X_2X_3+a_5X_3^2). \end{aligned} \tag{10}$$

The eigenvalues of the Jacobian matrix $\mathbf{TJT}^{-1}$, now in Jordan form, are $\lambda_{1/2}=\mu\pm\mathrm{i}w(w>0, \mu=0(r-r_T))$ and $\lambda_3<0$ with $\lambda_1+\lambda_2+\lambda_3=-(\sigma+b+1)$.

The coefficients of the normal form are given by the following equations:

$$\begin{aligned} &B=(\lambda_3-\mu)^2+w^2 \qquad C=(b+\mu)^2+w^2 \\ &D=C^{-1}(b+\mu)(b+\lambda_3) \qquad E=A/B \end{aligned} \tag{11}$$

$$a_1=2E[D(2\sigma-b)-\mu-1] \qquad a_2=2Ew(2\sigma-b)(b+\lambda_3)C^{-1}-2Ew \tag{12}$$

$$a_3=E(3\sigma+\lambda_3)+\tfrac{1}{2}a_1 \qquad a_4=\tfrac{1}{2}a_2 \qquad a_5=\tfrac{1}{2}(a_3-\tfrac{1}{2}a_1) \tag{13}$$

$$b_1=-\frac{2E}{w}\{(\sigma+\mu)(\lambda_3-\mu)+C(D-1)[1+(2\sigma-b)(b+\mu)C^{-1}]\} \tag{14}$$

$$b_2=-(a_1-4a_5) \qquad b_3=-E\left(w^{-1}(\sigma+\lambda_3)(\lambda_3-\mu)+Cw^{-1}\frac{2\sigma+\lambda_3}{b+\lambda_3}(D-1)\right)+\tfrac{1}{2}b_1 \tag{15}$$

$$b_4=\tfrac{1}{2}b_2 \qquad b_5=\tfrac{1}{2}(b_3-\tfrac{1}{2}b_1). \tag{16}$$

The integration of equation (4) making use of equations (1)-(3) gives

$$\begin{aligned} \partial_t\tilde{P}=\sum_{i=1,2}&\left(-\partial_{X_i}[(\dot{\tilde{X}}_i-(\mathbf{T}f)_i)\tilde{P}]-(\dot{\tilde{X}}_i-(\mathbf{T}f)_i)\frac{g_i}{2g}\tilde{P}\right) \\ &-(\dot{\tilde{X}}_3-(\mathbf{T}f)_3)\tilde{P}-2kg\tilde{P}+4kpg(\partial_{X_1}\tilde{F}_3+\partial_{X_2}\tilde{F}_3)\tilde{P} \\ &+kp^2\sum_{i=1,2}\left[\partial^2_{X_iX_i}\tilde{P}+\frac{g_i}{g}\partial_{X_i}\tilde{P}-\left(2g(\partial_{X_i}\tilde{F}_3)^2+\frac{g_i^2}{4g^2}\right)\tilde{P}\right] \end{aligned} \tag{17}$$

where $\dot{\tilde{X}}_i=\dot{X}_i(X_1, X_2, \tilde{F}_3)$, $i=1,2,3$, and $\tilde{C}_{33}=2k$, $\tilde{C}_{ii}=2pk$, $i=1,2$.

The CM equation for this particular problem can be calculated taking into account the statistical enslavement of X_3 to the order parameters valid in the regime under consideration, where there is a large separation of relaxation timescales. Writing the analytic expansion of X_3 as [8]

$$X_3=\tilde{F}_3(X_s)=\sum_{n=2}^{\infty}\left(\prod_{i+j=n} c_{ij}X_1^iX_2^j\right) \tag{18}$$

the CM coefficients can be calculated from the relation:

$$\dot{X}_3=\sum_{i=1,2}(\partial_{X_i}\tilde{F}_3)\dot{X}_i \tag{19}$$

making use of relations (10)-(16). Retaining only second-order terms, a simple calculation gives

$$c_{20}=(wc_{11}+2a_1)\lambda_3^{-1} \tag{20}$$

$$c_{11}=2\frac{\lambda_3a_2-2wa_1}{4w^2+\lambda_3^2} \tag{21}$$

$$c_{02}=\lambda_3^{-1}wc_{11}. \tag{22}$$

In order to display the relative size of the terms in (17) and obtain a FP equation for $\tilde{P}$ which corresponds to the CM-reduced equation, we must make use of the scaling relations and evaluate g explicitly.

The equation of continuity for $\tilde{P}$ is

$$\partial_t\tilde{P}=-\sum_{i=1,2}\partial_{X_i}[(\dot{X}_i-(\mathbf{T}f)_i)\tilde{P}]+\sum_{i,i'=1,2}\tilde{C}_{ii'}\partial^2_{X_iX_{i'}}\tilde{P}. \tag{23}$$

We introduce the following scaling:

$$k=\mathrm{O}((r-r_T)^2) \qquad d=(2g)^{-1/2}=\mathrm{O}((r-r_T)) \qquad X_1,X_2=\mathrm{O}((r-r_T)^{1/2}). \tag{24}$$

Then, to order $(r-r_T)$, (17) reduces to (23) if and only if

$$g_0=\lambda_3/2k \qquad g_1=\frac{a_3}{k}-\frac{4p(a_1+\frac{1}{2}a_2)}{k} \qquad g_2=\frac{a_4}{k}-\frac{2pa_2}{k}. \tag{25}$$

The Gaussian width d provides a measure of the local attractivity of the Lorenz invariant surface [9]. Therefore, as we take the limit $r\to r_T+0$, the probability distribution tends to a Dirac delta function peaked at the CM. Thus, the CM becomes less attractive as we depart from criticality. The scaling relations given in (24) determine the scaling of the intrinsic fluctuations.

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